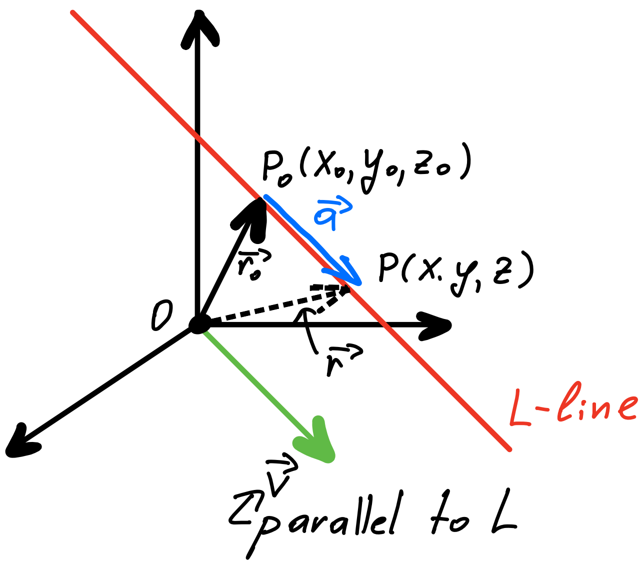
Review

Lines

$$\boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$$ - vector equation of L

$t \in \mathbb{R}$ parameter



• Let $\vec{v} = \langle a, b, c \rangle$

then in components:

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

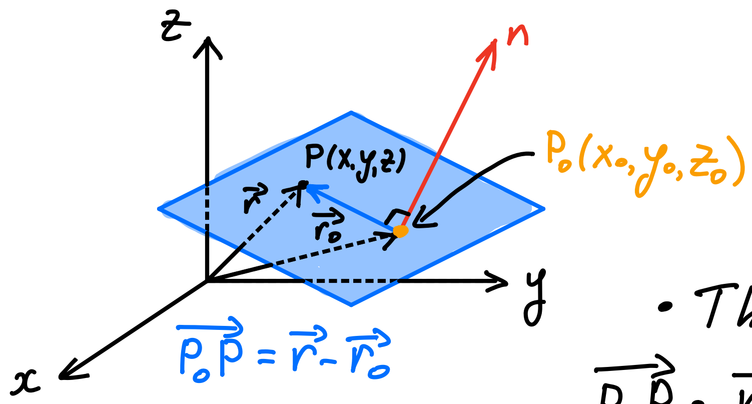
$$\boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}}$$

- parametric equations of the line L through point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$

$(t =)$ $$\boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$ - symmetric equations of L

Planes

- A plane is defined by a point P_0 and an orthogonal vector $\vec{n}$ ("the normal vector")



$\overrightarrow{P_0P} = \vec{r} - \vec{r_0}$

- Then for any point P on the plane,

$\overrightarrow{P_0P} \cdot \vec{n} = 0 \Rightarrow \boxed{\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0}$

or $\boxed{\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}}$

vector eq. of the plane

Reminder:

$\vec{a} = \langle a_1, a_2, a_3 \rangle \qquad \vec{b} = \langle b_1, b_2, b_3 \rangle$

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

vectors are orthogonal $\Leftrightarrow$ $\vec{a} \cdot \vec{b} = 0$

If $\vec{n} = \langle a, b, c \rangle$, $\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, then eq. of the plane is

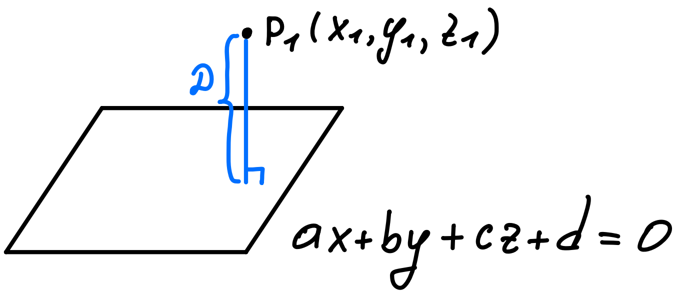
$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

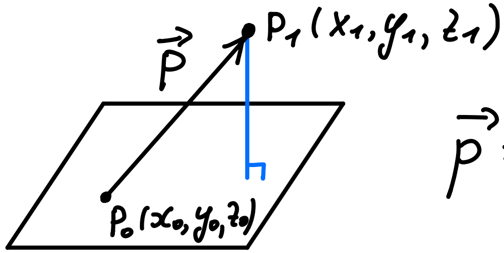
- scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$

Distances

Distance from a point to a plane.



Let P_0 be any point in the plane



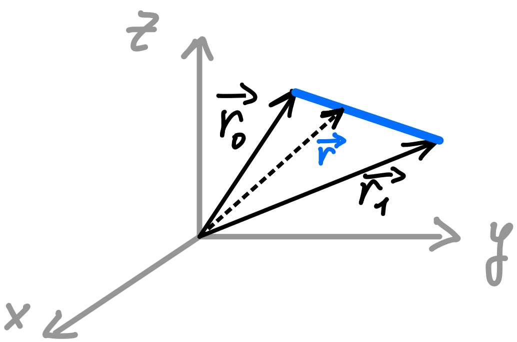
$\vec{p} = \overrightarrow{P_0P_1}$

$\vec{p} = \langle x_1 - x_0, y_1 - y_0, z_1 - z_0 \rangle$

$$\boxed{\text{Distance } D} = |\text{comp}_{\vec{n}} \overrightarrow{P_0P_1}| = \frac{|\vec{n} \cdot \vec{p}|}{|\vec{n}|} = \frac{|a(x_1 - x_0) + b(y_1 - y_0) + c(z_1 - z_0)|}{\sqrt{a^2+b^2+c^2}}$$

$$\boxed{= \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}}$$

Line segment: line segment from $\vec{r_0}$ to $\vec{r_1}$ is given by



$$\vec{r} = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) = (1-t)\vec{r_0} + t\vec{r_1} \quad \text{with } 0 \le t \le 1$$

Quizz

Quizz

1) Find an equation of the line trough P_0 parallel to vector $\vec{v}$

(a) $P_0(3, 1, -2)$ $\quad$ $\vec{v} = (5, -3, 1)$

(b) $P_0(0, 4, 7)$ $\quad$ $\vec{v} = (3, 4, -2)$

(c) $P_0(1, 1, 7)$ $\quad$ $\vec{v} = (0, 1, -2)$

2) Find an equation of the plane trough P_0 with normal vector $\vec{n}$

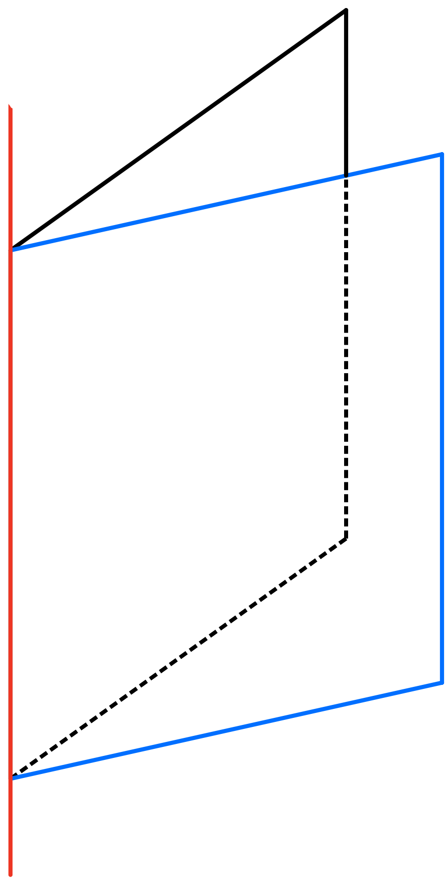
(a) $P_0(-2, 7, 7)$ $\quad$ $\vec{n} = (5, 3, 0)$

(b) $P_0(1, 4, 1)$ $\quad$ $\vec{n} = (-3, 4, 1)$

(c) $P_0(3, 0, 1)$ $\quad$ $\vec{n} = (1, -2, 2)$

3) Two non-parallel planes have normal vectors $\vec{n}_1$ and $\vec{n}_2$.

Find the direction vector of the intersecting line.



Quiz Answers

(1) $P_0(3, 1, -2)$ $\vec{v} = (5, -3, 1)$

$$L: \begin{cases} x = 3 + 5t \\ y = 1 - 3t \\ z = -2 + 1t \end{cases}$$

parametric equations

(2) $P_0(-2, 7, 7)$ $\vec{n} = (5, 3, 0)$

$$5(x-(-2)) + 3(y-7) + 0(z-7) = 0$$

$\Longleftrightarrow$ $5(x+2) + 3(y-7) = 0$ - scalar equation

(3) The direction vector of the intersect. line is orthogonal to both $\vec{n_1}$ and $\vec{n_2}$. So, we can take $\vec{v} = \vec{n_1} \times \vec{n_2}$

$P_0(3, 1, -2)$, $\vec{v} = (5, -3, 1)$

$$\frac{x-3}{5} = \frac{y-1}{-3} = \frac{z-(-2)}{1}$$

- symmetric equation

$P_0(-2, 7, 7)$, $\vec{n} = (5, 3, 0)$

$$5(x-(-2)) + 3(y-7) + 0(z-7) = 0$$

$$5(x+2) + 3(y-7) = 0$$

- scalar equation

$P_0(0, 4, 7)$, $\vec{v} = (3, 4, -2)$

$$\begin{cases} x = 0 + 3t \\ y = 4 + 4t \\ z = 7 - 2t \end{cases}$$

parametric equations

$P_0(1, 4, 1)$, $\vec{n} = (-3, 4, 1)$

$$-3(x-1) + 4(y-4) + (z-1) = 0$$

OR

$$-3x + 4y + z - 14 = 0$$

$P_0(1, 1, 7)$, $\vec{v} = (0, 1, -2)$

$$\vec{r} = \langle 1, 1, 7 \rangle + \langle 0, t, -2t \rangle$$

- vector equation

$P_0(3, 0, 1)$, $\vec{n} = (1, -2, 2)$

$$(x-3) - 2y + 2(z-1) = 0$$

OR

$$x - 2y + 2z - 5 = 0$$

problems

1) Find symmetriq equations of the line going through the points $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$.

2) $A(1,0,2)$ $B(3,6,6)$

a) Find param. and sym. equations of the line L through the points A and B

b) Where does L intersect xy-plane?

3) Describe the line segment from $A(1,0,2)$ to $B(2,3,4)$ in terms of parametric and vector equations.

4) $P(-1,2,5)$, $Q(0,4,8)$, $R(1,2,0)$

a) Find an equation of the plane through P, Q, R

b) Find the intersection point of this plane with the line

$$x=-t,\ y=t-1,\ z=6t-2$$

5) Find a sym. eq. of the intersect. line L

of two planes $x+y+2z=0$, $-y+z=1$

Solutions

(1) Line through $P_0(x_0, y_0, z_0)$, $P_1(x_1, y_1, z_1)$ has direction numbers $x_1 - x_0, y_1 - y_0, z_1 - z_0$

$\Rightarrow$ sym. eq. $\dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0} = \dfrac{z - z_0}{z_1 - z_0}$

(2) $P_0 = A = (1, 0, 2)$ $\qquad \vec{v} = \overrightarrow{AB} = \langle 3-1, 6-0, 6-2 \rangle = \langle 2, 6, 4 \rangle$

a) $\Rightarrow$ param. eq. $\begin{cases} x = 1 + 2t \\ y = 0 + 6t \\ z = 2 + 4t \end{cases}$ AND $\dfrac{x-1}{2} = \dfrac{y}{6} = \dfrac{z-2}{4}$ – sym. eq.

b) xy-plane is given by $z = 0$.

So we are looking for a point (x, y, z) which satisfies both $\underbrace{z = 0,}$ and $\dfrac{x-1}{2} = \dfrac{y}{6} = \dfrac{z-2}{4}$.

$\Rightarrow (x, y, 0)$ such that $\underbrace{\dfrac{x-1}{2} = \dfrac{y}{6} = \dfrac{0-2}{4} = -\dfrac{1}{2}}$

Multiply by 2: $x - 1 = \dfrac{y}{3} = -1$

$\Rightarrow x = 0, \; y = -3$

The intersection point is $(0, -3, 0)$

(3) line segment from $A(1,0,2)$ to $B(2,3,4)$ is given by

$$x = 1+t,\quad y = 3t,\quad z = 2+2t \quad \text{with} \quad 0 \le t \le 1$$

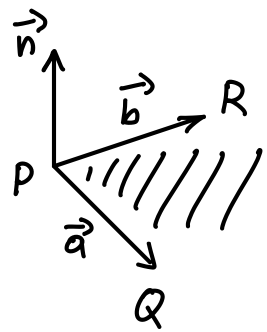
or by the corresponding vector equation

$$r(t) = \langle 1+t,\ 3t,\ 2+2t \rangle \quad \text{with} \quad 0 \le t \le 1$$

(4) (a) $\vec{a} = \overrightarrow{PQ} = \langle 1, 2, 3 \rangle \qquad \vec{b} = \overrightarrow{PR} = \langle 2, 0, -5 \rangle$

normal vector: $\vec{n} = \vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 0 & -5 \end{vmatrix} = \langle 2(-5) - 0(3),\ 3(2) - 1(-5),\ 1(0) - 2(2) \rangle$

$$= \langle -10, 11, -4 \rangle$$



Take $P_0 = P$, then eq. of the plane:

$$-10(x+1) + 11(y-2) - 4(z-5) = 0$$

or $$-10x + 11y - 4z - 12 = 0$$

(b) $$-10(-t) + 11(t-1) - 4(6t-2) - 12 = 0$$

$$-3t - 11 + 8 - 12 = 0$$

$$3t = -15 \Rightarrow t = -5$$

$$\boxed{(x,y,z) = (5,-6,-32)}$$

<u>Check:</u>

$$\underbrace{-10(5) + 11(-6)}_{-116}\ \underbrace{-4(-32)}_{+128} - 12 = 0$$

$$-12 = 0$$

(5) The normal vectors are

$\vec{n}_1 = \langle 1, 1, 2 \rangle$ $\quad (x + y + 2z = 0)$

$\vec{n}_2 = \langle 0, -1, 1 \rangle$ $\quad (-y + z = 1)$

Need a point on L: set $z = 0$

$$\begin{cases} x + y + 2(0) = 0 \\ -y + 0 = 1 \end{cases} \Rightarrow \begin{array}{l} y = -1 \\ x = 1 \end{array} \Rightarrow$$

$P_0(1, -1, 0)$

Direction vector $\vec{v}$ is orthogonal to both $\vec{n}_1$ and $\vec{n}_2$, take

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ 0 & -1 & 1 \end{vmatrix} = (1(1) - 2(-1))\vec{i} + (2(0) - 1(1))\vec{j} + (1(-1) - 1(0))\vec{k}$$

$$= 3\vec{i} - \vec{j} - \vec{k} = \langle 3, -1, -1 \rangle$$

Sym. eq. for L: $\quad \dfrac{x-1}{3} = \dfrac{y+1}{-1} = \dfrac{z}{-1}$

<u>Rmk:</u> We can obtain same line as intersection of planes

$$\frac{x-1}{3} = \frac{y+1}{-1} \quad \text{and} \quad \frac{y+1}{-1} = \frac{z}{-1}$$